Scattering amplitudes at high energies and anomalous dimensions of local operators in QCD and in supersymmetric models

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Content

1. Gluon reggeization
2. BFKL approach
3. Integrability of the BKP equations
4. Pomeron in $N = 4$ SUSY
5. BDS amplitudes at large energies
6. Absence of the Regge factorization
7. Mandelstam cuts
8. Open integrable Heisenberg spin chain

1 Gluon reggeization

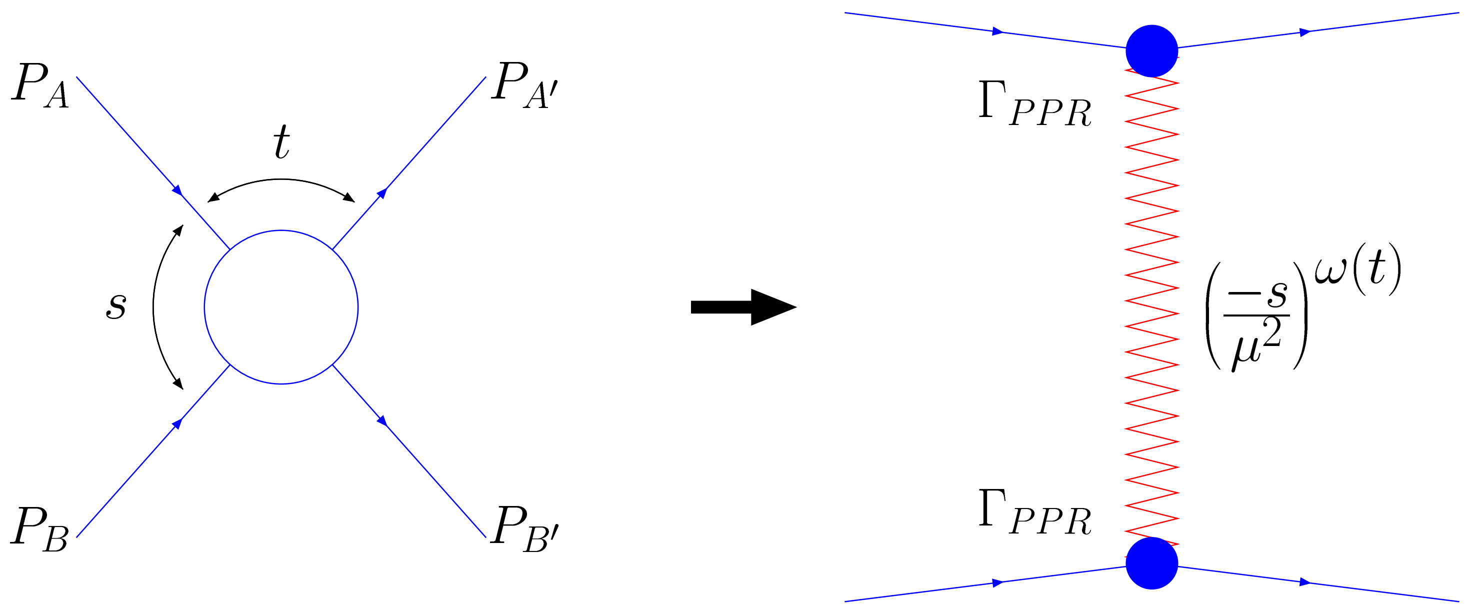


Figure 1: Elastic amplitude in the Regge kinematics

$$M = 2s\, g\, T^c_{A'A}\, \delta_{\lambda_{A'}\lambda_A}\, \frac{s^{\omega(t)}}{t}\, g\, T^c_{B'B}\, \delta_{\lambda_{B'}\lambda_B}\ ,\ \ \omega_{LLA}(-|q|^2) = -\frac{\alpha_s N_c}{2\pi}\ \ln\frac{|q^2|}{\lambda^2}$$

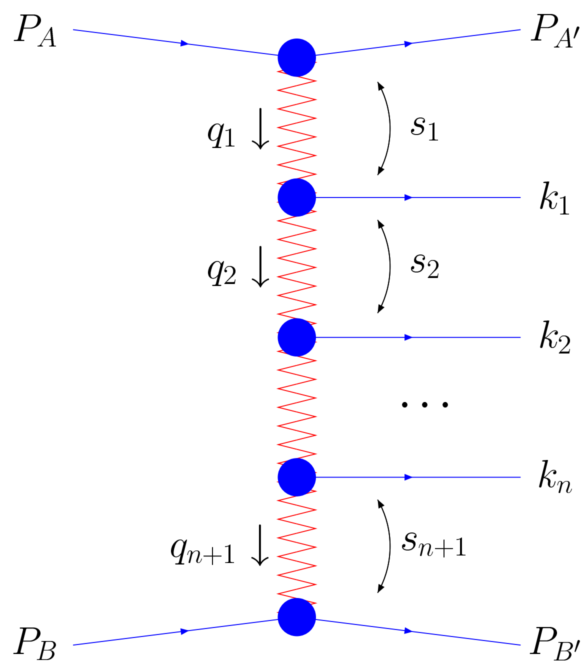


Figure 2: Multi-Regge amplitude

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, g T^{d_1}_{c_2 c_1} C(q_2, q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} \ldots C(q_n, q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2},$$

$$C(q_2, q_1) = \frac{q_2\, q_1^*}{q_2^* - q_1^*}\,,\ \ \sigma_t = \sum_n \int d\Gamma_n\, |M_{2\to 1+n}|^2$$

2 BFKL equation (1975)

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1,\vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1,\vec{\rho}_2)\;,\;\;\sigma_t \sim s^{\Delta}\,,\;\;\Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2)p_1 p_2^* + \frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2)p_1^* p_2 - 4\psi(1)$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,;\;\; m = \gamma + n/2\,,\;\; \widetilde{m} = \gamma - n/2\,,\;\; \gamma = 1/2 + i\nu$$

Pomeron energy and intercept

$$E_{LLA}(m,\widetilde{m}) = \epsilon(m) + \epsilon(\widetilde{m})\,,\;\; \epsilon(m) = \psi(m) + \psi(1-m) - 2\psi(1)\,,$$

$$\Delta_{LLA} = \frac{4\alpha}{\pi}N_c > 0$$

3 BKP equation (1980)

Bartels-Kwiecinski-Praszalowicz equation

$$E\,\Psi(\vec\rho_1,...,\vec\rho_n) = H\,\Psi(\vec\rho_1,...,\vec\rho_n)\;,\;\; H = \sum_{k<l} \frac{\vec T_k \vec T_l}{-N_c}\, H_{kl}$$

Holomorphic separability at large N_c (L. (1988))

$$H = \frac{1}{2}\,(h + h^*)\;,\;\; [h, h^*] = 0\;,\;\; h = \sum_{k=1}^{n} h_{k,k+1}\;,$$

$$h_{12} = \ln p_1 + \ln p_2 + \frac{1}{p_1}(\ln\rho_{12})\,p_1 + \frac{1}{p_2}(\ln\rho_{12})\,p_2 - 2\psi(1)$$

Holomorphic factorization of wave functions

$$\Psi(\vec\rho_1, \vec\rho_2, ..., \vec\rho_n) = \sum_{r,s} a_{r,s}\,\Psi_r(\rho_1, ..., \rho_n)\,\Psi_s(\rho_1^*, ..., \rho_n^*)$$

4 Integrability at $N_c \to \infty$ (L. (1993))

Monodromy and transfer matrices

$$t(u) = L_1(u)L_2(u)...L_n(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \; T(u) = A(u) + D(u)\,,$$

$$[T(u), T(v)] = [T(u), h] = 0\,, \; L_k(u) = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix}$$

Yang-Baxter equation

$$t^{s_1}_{r'_1}(u)\, t^{s_2}_{r'_2}(v)\, l^{r'_1 r'_2}_{r_1 r_2}(v-u) = l^{s_1 s_2}_{s'_1 s'_2}(v-u)\, t^{s'_2}_{r_2}(v)\, t^{s'_1}_{r_1}(u)\,, \; \hat{l} = u\,\hat{1} + i\,\hat{P}$$

Duality symmetry (L. (1999))

$$p_r \to \rho_{r+1,r} \to p_{r+1}$$

Integrable Heisenberg spin chain (L. (1994); F.,K. (1995))

5 Pomeron in $N = 4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\;\hat{a} = g^2 N_c/(16\pi^2)\,,$$

Hermitian separability in $N = 4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\;\beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right]$$

Maximal transcendentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\;-\;\frac{\Psi(k+1) - \Psi(1)}{k+M}\right)$$

6 Maximal helicity violation

BDS amplitudes in $N=4$ SUSY at $N_c \gg 1$ (2005)

$$A^{a_1,\dots,a_n} = \sum_{\{i_1,\dots,i_n\}} Tr\, T^{a_{i_1}} T^{a_{i_2}} \dots T^{a_{i_n}}\, f(p_{i_1}, p_{i_2}, \dots, p_{i_n})\,, \;\; f = f_B\, M_n$$

Invariant amplitudes

$$\ln M_n = \sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) \left(-\frac{1}{2\epsilon^2} \sum_{i=1}^{n} \left(\frac{\mu^2}{-s_{i,i+1}} \right)^{\epsilon} + F_n^{(1)}(0) \right) + C^{(l)} \right)\,,$$

$$a = \frac{\alpha N_c}{2\pi} \left(4\pi e^{-\gamma}\right)^{\epsilon}\,,\;\; C^{(1)} = 0\,,\;\; C^{(2)} = -\zeta_2^2/2\,,\;\; f^{(l)}(\epsilon) = \sum_{k=0}^{2} \epsilon^k\, f_k^{(1)}$$

Cusp anomalous dimension

$$f_0^{(l)} = \frac{1}{4}\gamma_K^{(l)}\,,\;\; f_1 = \beta(f) = -a\zeta_3/2 + a^2(2\zeta_5 + 5\zeta_2\zeta_3/3) + \dots$$

7 Elastic BDS amplitude

Regge asymptotics at $s/t \to \infty$

$$M_{2\to 2} = \Gamma(t) \left(\frac{-s}{\mu^2}\right)^{\omega(t)} \Gamma(t)\,, \;\; a = \frac{g^2\, N_c}{8\pi^2}\, \left(4\pi\, e^{-\gamma}\right)^{\epsilon}$$

Reggeized gluon trajectory

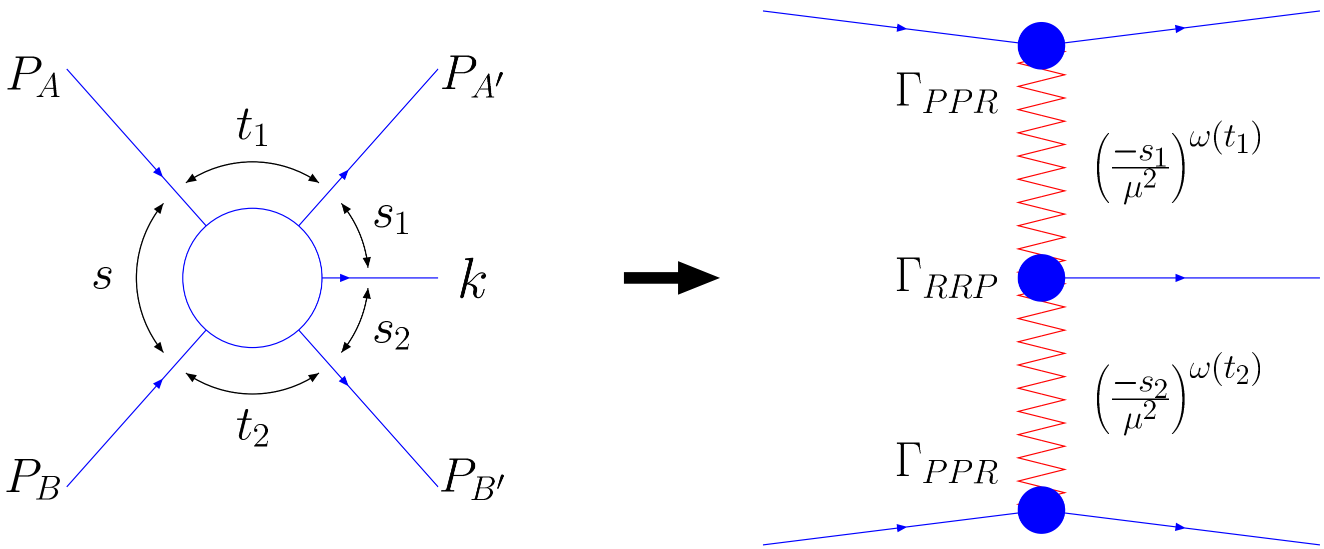
$$\omega(t) = -\frac{\gamma_K(a)}{4} \ln \frac{-t}{\mu^2} + \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{4\epsilon} + \beta(a')\right)$$

Reggeon residues

$$\ln \Gamma(t) = \ln \frac{-t}{\mu^2} \int_0^a \frac{da'}{a'} \left(\frac{\gamma_K(a')}{8\epsilon} + \frac{\beta(a')}{2}\right) + \frac{C(a)}{2} + \frac{\gamma_K(a)}{2}\,\zeta_2$$

$$- \int_0^a \frac{da'}{a'} \ln \frac{a}{a'} \left(\frac{\gamma_K(a')}{4\epsilon^2} + \frac{\beta(a')}{\epsilon} + \delta(a')\right)$$

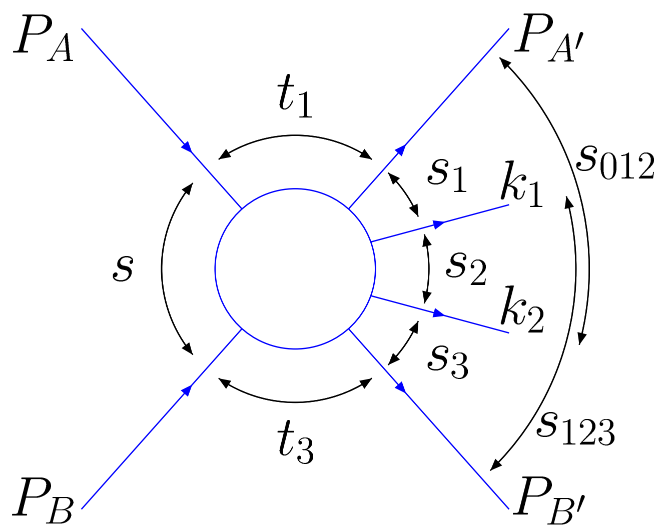
8 One particle production



$$\ln \Gamma_{\kappa=s_1 s_2/s} = -\frac{1}{2}\left(\omega(t_1)+\omega(t_2)-\int_0^a \frac{da'}{a'}\left(\frac{\gamma_K(a')}{4\epsilon}+\beta(a')\right)\right)\ln\frac{-\kappa}{\mu^2}-$$

$$\frac{\gamma_K(a)}{16}\left(\ln^2\frac{-\kappa}{\mu^2}-\ln^2\frac{-t_1}{-t_2}-\zeta_2\right)-\frac{1}{2}\int_0^a \frac{da'}{a'}\ln\frac{a}{a'}\left(\frac{\gamma_K(a')}{4\epsilon^2}+\frac{\beta(a')}{\epsilon}+\delta(a')\right)$$

9 Regge factorization violation



$$M_{2\to 4}|_{s_2>0;\, s_1,s_3<0} = \exp\left[\frac{\gamma_K(a)}{4}\, i\pi\, \left(\ln\frac{t_1 t_2}{(\vec{k}_1+\vec{k}_2)^2\mu^2} - \frac{1}{\epsilon}\right)\right]$$

$$\times\Gamma(t_1)\left(\frac{-s_1}{\mu^2}\right)^{\omega(t_1)}\Gamma(t_2,t_1)\left(\frac{-s_2}{\mu^2}\right)^{\omega(t_2)}\Gamma(t_3,t_2)\left(\frac{-s_3}{\mu^2}\right)^{\omega(t_3)}\Gamma(t_3)$$

10 Mandelstam cuts in j_2-plane

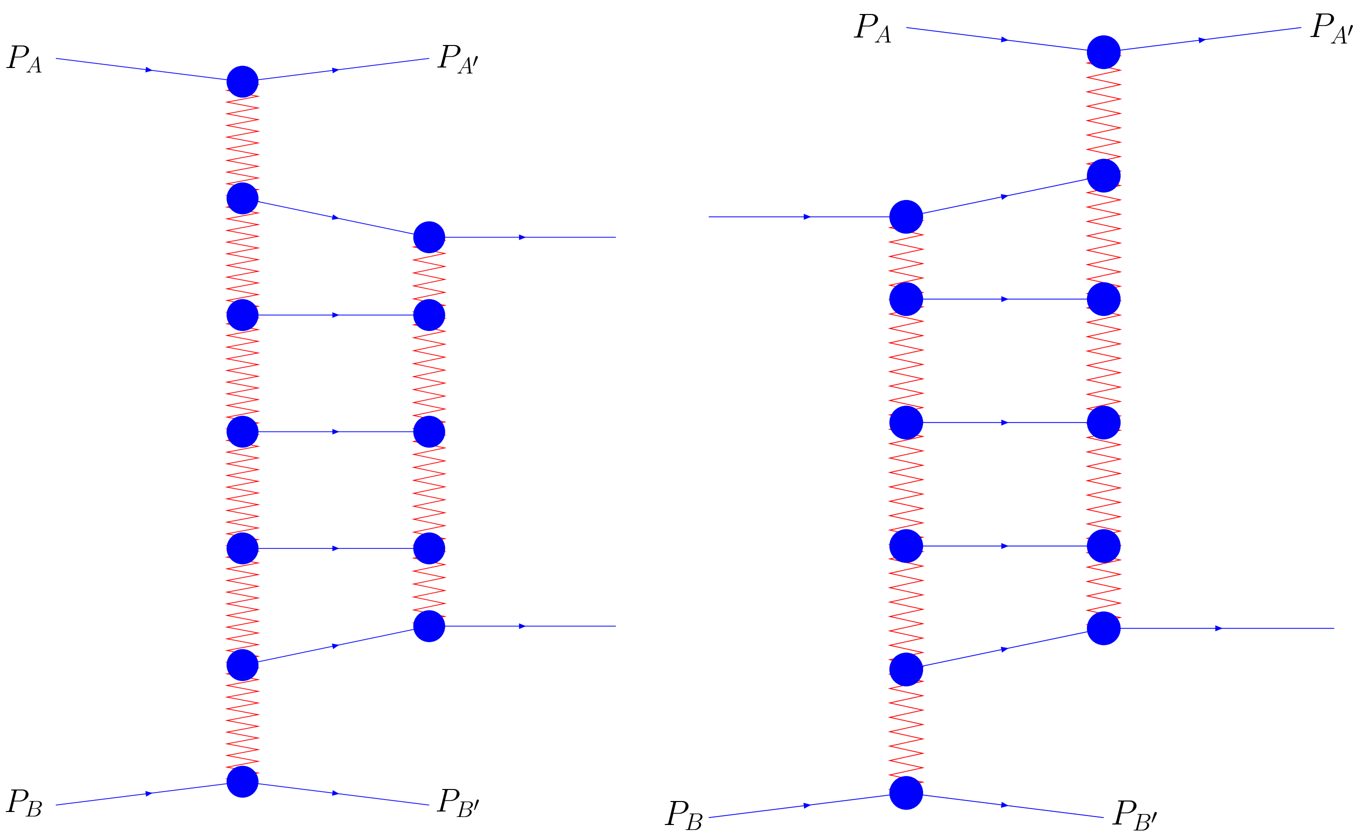


Figure 3: BFKL ladders in $M_{2\to 4}$ and $M_{3\to 3}$

11 Multi-gluon reggeon states

Regge trajectories of the octet composite states

$$\Delta_n = -a\left(\epsilon + \epsilon^* + \ln\frac{t}{\mu^2} - \frac{1}{\epsilon}\right)$$

Holomorphic hamiltonian for n-gluon composite states

$$h = \ln(Z_1^2\partial_1) - 2\psi(1) + \ln\partial_{n-1} + \sum_{k=1}^{n-2} h_{k,k+1}\,,\ p_k = Z_{k-1,k}\,,\ Z_0 = 0\,,\ Z_n = \infty$$

Pair hamiltonian of the spin chain

$$h_{1,2} = \ln(Z_{12}^2\partial_1) + \ln(Z_{12}^2\partial_2) - 2\ln Z_{12} - 2\psi(1)$$

Integrals of motion and Baxter equation for the open spin chain

$$[D(u), h] = 0\,,\ D(u)Q(u) = (u-i)^{n-1}Q(u-i)$$

12 Two and three gluon composite states

Two gluon eigenfunctions and eigenvalues

$$\Psi_{n,\nu} = \left(\frac{p_1}{p_2}\right)^{i\nu+n/2} \left(\frac{p_1^*}{p_2^*}\right)^{i\nu-n/2}, \; E_{n,\nu} = 2Re\,\psi(i\nu + \frac{|n|}{2}) - 2\psi(1)$$

Baxter-Sklyanin representation for 3-gluon state

$$\Psi^t(\vec{p_1}, \vec{p_2}) = P^{-a_1-a_2}\,(P^*)^{-\widetilde{a}_1-\widetilde{a}_2} \int d\nu \sum_n u\,\widetilde{u}\,Q(u,\widetilde{u}) \left(\frac{p_1}{p_2}\right)^u \left(\frac{p_1^*}{p_2^*}\right)^{u^*}$$

Baxter function

$$Q(u,\widetilde{u}) = \frac{\Gamma(-u)\,\Gamma(-\widetilde{u})}{\Gamma(1+u)\,\Gamma(1+\widetilde{u})}\,\frac{\Gamma(u-a_1)\,\Gamma(u-a_2)}{\Gamma(1-\widetilde{u}+\widetilde{a}_1)\,\Gamma(1-\widetilde{u}+\widetilde{a}_2)}, \; u = i\nu + \frac{n}{2}$$

Wave function in the coordinate representation

$$\Psi = Z_2^{a_1+a_2}(Z_2^*)^{\widetilde{a}_1+\widetilde{a}_2} \int \frac{d^2y}{|y|^2}\, y^{-a_2}\,(y^*)^{\widetilde{a}_2} \left(\frac{y-1}{y-Z_2/Z_1}\right)^{a_1} \left(\frac{y^*-1}{y^*-Z_2^*/Z_1^*}\right)^{\widetilde{a}_1}$$

13 Anomalous dimensions

Wilson twist-2 operators

$$\mathcal{O}^{g}_{\mu_1,\ldots,\mu_j} = \hat{S} G^{a}_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} G^{a}_{\rho\mu_j} \,,$$

$$\tilde{\mathcal{O}}^{g}_{\mu_1,\ldots,\mu_j} = \hat{S} G^{a}_{\rho\mu_1} D_{\mu_2} D_{\mu_3} \ldots D_{\mu_{j-1}} \tilde{G}^{a}_{\rho\mu_j} \,,$$

$$\mathcal{O}^{q}_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^{a} \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^{a} \,,$$

$$\tilde{\mathcal{O}}^{q}_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Psi}^{a} \gamma_5 \gamma_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Psi^{a} \,,$$

$$\mathcal{O}^{\varphi}_{\mu_1,\ldots,\mu_j} = \hat{S} \bar{\Phi}^{a} D_{\mu_1} D_{\mu_2} \ldots D_{\mu_j} \Phi^{a}$$

Diagonalization of γ in the Born Approximation

$$\begin{vmatrix} 8S_1(j-2) & 0 & 0 \\ 0 & 8S_1(j) & 0 \\ 0 & 0 & 8S_1(j+2) \end{vmatrix} , \begin{vmatrix} 8S_1(j-1) & 0 \\ 0 & 8S_1(j+1) \end{vmatrix}$$

14 Universal anomalous dimension

Most transcendental functions (A.K.,L.L. (2000))

$$\gamma_{uni}(j) = \hat{a}\gamma^{(0)}_{uni}(j) + \hat{a}^2\gamma^{(1)}_{uni}(j) + \hat{a}^3\gamma^{(2)}_{uni}(j) + \dots ,$$

$$\gamma^{(1)}_{uni}(j+2)/8 = 2S_1(j)\left(S_2(j) + S_{-2}(j)\right) - 2S_{-2,1}(j) + S_3(j)$$

$$\gamma^{(2)}_{uni}(j+2)/32 = -12\,(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}) - 3\,S_{-5} - 2\,S_3\,S_{-2} - S_5$$
$$+6\,(S_{-4,1} + S_{-3,2} + S_{-2,3}) - 2\,S_1^2\,(3\,S_{-3} + S_3 - 2\,S_{-2,1}) -$$
$$S_2\,(S_{-3} + S_3 - 2\,S_{-2,1}) - S_1\left(8S_{-4} + S_{-2}^2 + 4S_2S_{-2} + 2S_2^2\right)$$
$$+24\,S_{-2,1,1,1} - S_1\,(3S_4 - 12S_{-3,1} - 10S_{-2,2} + 16S_{-2,1,1})$$

Harmonic sums

$$S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\, S_{b,c,\cdots}(m)\,,\ ,\ S_{-a,b,c\cdots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\, S_{b,c,\cdots}(m)\Big)$$

Comparison with other approaches (A.K.,L.L.,A.O.,V.V.)

Singularities at $j = 1 + \omega \to 0$

$$\gamma_{uni}^{N=4}(j) = \hat{a}\,\frac{4}{\omega} - 32\zeta_3\,\hat{a}^2 + 32\zeta_3\,\hat{a}^3\,\frac{1}{\omega} + \ldots$$

DL resummation at $j + 2r = \omega \to 0$

$$\gamma_{uni} = 4\,\frac{\hat{a}}{\omega} + \frac{\gamma_{uni}^2}{\omega}$$

Anomalous dimensions at large j

$$\gamma_{uni} = a(z)\,\ln j\,,\;\; z = \frac{\alpha N_c}{\pi} = 4\hat{a}$$

Perturbative results

$$a = -z + \frac{\pi^2}{12}\,z^2 - \frac{11}{720}\pi^4 z^3 + \ldots$$

Polyakov AdS/CFT prediction

$$\lim_{z \to \infty} a = -z^{1/2} + \frac{3 \ln 2}{4\pi} + \ldots$$

Resummation

$$\widetilde{a} = -z + \frac{\pi^2}{12}\,\widetilde{a}^2 = -z + \frac{\pi^2}{12}\,z^2 - \frac{1}{72}\pi^4 z^3 + \ldots$$

15 Discussion

1. Gluon reggeization.
2. Integrability of BFKL dynamics in LLA.
3. Remarkable properties of NLLA in $N = 4$ SUSY.
4. BDS amplitudes in the multi-Regge kinematics.
5. Breakdown of the Regge factorization.
6. Mandelstam cuts in the planar amplitudes M_n for $n > 5$.
7. Integrable open spin chain for scattering amplitudes.