One-Loop Computations in Lattice Perturbation Theory

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Outline

Why lattice perturbation theory is needed?

Illustrative Example: Weak Expansion in the ϕ^3 Theory

One-Loop Boson Integrals at Zero External Momentum

Fermion Integrals

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Where lattice perturbative calculations are either useful or necessary?

- To determine the Λ parameter of QCD in the lattice regularization and its relation to the respective value Λ_{QCD} in the continuum theory.

$$\begin{aligned}\Lambda \;=\; & \lim_{a\to 0} a^{-1}\cdot\left(b_0 g^2(a)\right)^{-\frac{b_1}{2b_0^2}}\cdot e^{-\frac{1}{2b_0 g^2(a)}} \qquad (1)\\ & \exp\left\{-\int_0^{g(a)} dt\left(\frac{1}{\beta(t)}+\frac{1}{b_0 t^3}-\frac{b_1}{b_0^2 t}\right)\right\},\end{aligned}$$

where b_i are the coefficients of the expansion

$$\beta(g) = -g^3\left[b_0 + b_1 g^2 + b_2 g^4 + \dots\right] \qquad (g\to 0). \qquad (2)$$

and $g(a)$ is the solution of the RG equation $a\,\dfrac{dg}{da} = \beta(g)$.

The relation between the bare lattice and the renormalized coupling (defined as the three-point function at a certain momentum p) is

$$g_R(p) = g(a)\Big[1+g^2(a)\Big(-b_0 \log ap + C^L + O(a^2p^2 \log ap)\Big) + O(g^4)\Big],$$

whereas for the continuum coupling ($\bar{MS}$ scheme) one has

$$g_R(p) = g_{ms}\Big[1 + g_{ms}^2\Big(- b_0 \log \frac{p}{\mu} + C^{ms}\Big) + O(g_{ms}^4)\Big].$$

Combining these two equations, we arrive at

$$g(a) = g_{ms}\Big[1 + g^2(a)\Big(C^{ms} - C^L + b_0 \log a\mu\Big) + O(g^4) + O(a^2)\Big].$$

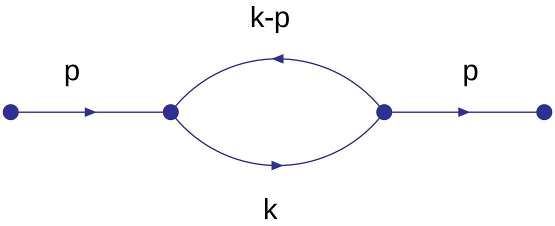
Substituting this relation into the definition of Λ, we arrive at the relation between Λ_{QCD} and Λ_{LAT}. For the pure gauge Wilson action (covariant gauge) one obtains

$$\frac{\Lambda_{ms}}{\Lambda_{\mathrm{l}at}} = 28.80934(1). \qquad (3)$$

- A determination of the renormalization factors of matrix elements of operators and of the renormalization of the bare parameters of the Lagrangian, like couplings and masses. Perturbation theory is needed to establish the connection of the matrix elements simulated on a lattice with their values in the physical continuum theory. Every lattice action defines a specific regularization scheme, and thus one needs a complete set of renormalization computations in order for the results obtained in Monte Carlo simulations be understood properly.

- Studies of the anomalies on the lattice. Perturbation theory is also important for defining chiral gauge theories on the lattice at all orders in the gauge coupling. Thus the lattice is the only regularization that can preserve both chiral and gauge invariance (without destroying basic features like locality and unitarity).
- study of the general approach to the continuum limit, including the recovery of the continuum symmetries broken by the lattice regularization (like Lorentz or chiral symmetry) in the limit $a \to 0$, and the scaling violations, i.e., the corrections to the continuum limit which are of order a^n.

- Perturbative calculations provide the only possibility for an analytical control over the continuum limit. Lattice perturbation theory is tightly connected to the continuum limit of lattice QCD. Because of asymptotic freedom, one has $g_0 \to 0$ when $\mu \to \infty$, which means $a \to 0$. Perturbative calculations play an important role in proving the renormalizability of lattice gauge theories.



$$= \frac{G_0^2(p)}{(2\pi)^4} \int_{BZ} dk$$

$$\frac{1}{\left(m^2 + \frac{2}{a^2}\sum_{\mu=1}^{4}(1-\cos(k_\mu a))\right)\left(m^2 + \frac{2}{a^2}\sum_{\mu=1}^{4}(1-\cos((p-k)_\mu a))\right)}$$

This integral is extremely complicated and cannot be calculated analytically at finite values of a. In the limit $a \to 0$ it can be evaluated using the

Kawai–Nakayama–Seo method.

The above integrand can be represented in the form

$$I(k,p,m^2;a) = I(k,0,0;a) \; + \; (I(k,p,m^2;a) - I(k,0,0;a)), \quad (4)$$

- $I(k,p,m^2;a) - I(k,0,0;a)$ has a smooth continuum limit ($pa \to 0$ and $ma \to 0$) and involves no UV divergencies. **It can be computed with some continuum regularization**
  such as dimensional or with fictitious mass.
- Both $I(k,0,0;a)$ and $(I(k,p,m^2;a) - I(k,0,0;a))$ involve IR divergencies.

Though these IR divergencies cancel each other, IR regularization is needed:

$$I(k,p,m^2;a) = \lim_{\mu_R^2 \to 0} I(k,p,m^2;a,\mu_R^2).$$

Thus the sought-for integral for the vacuum polarization, is represented as the sum of the integral over the Euclidean momentum space

$$\Pi_C(p) = \lim_{\mu_R^2 \to 0} \int dk \left(\frac{1}{(k^2 + m^2 + \mu_R^2)((k-p)^2 + m^2 + \mu_R^2)} - \frac{1}{(k^2 + \mu_R^2)^2} \right)$$

and the "zero-momentum" integral over the Brillouin zone

$$\Pi_{latt}(p) = \lim_{\mu_R^2 \to 0} \int_{BZ} \frac{dk}{\left(\mu_R^2 + \frac{2}{a^2} \sum_{\mu=1}^{4} (1 - \cos(p_\mu a)) \right)^2}.$$

Gauge Theories

The action in a specific gauge has the form

$$S = S_{phys} + S_{gf} + S_{FP} + S_{measure} + S_{Sym} \qquad (5)$$

where $S_{phys} = S_W$ or $S_{overlap}$ or S_{SW} etc. and the gauge fixing term in the general covariant gauge is given by

$$S_{gf} = \frac{1}{(1-\xi)} \sum_{x,a,\mu} \left(\hat{\partial}_\mu^B A_\mu^a(x) \right)^2 . \qquad (6)$$

where

$$\partial_\mu^B \phi(x) = \frac{\phi(x) - \phi(x-\hat{\mu})}{a}.$$

S_{gf} and the quadratic part of $S_{phys} = S_W$ determine the lattice gluon propagator

$$\mathcal{P}(k) = \frac{\delta_{\mu\nu}}{\hat{k}^2} - \xi \, \frac{\hat{k}_\mu \hat{k}_\nu}{\hat{k}^4} \qquad \text{where} \qquad \hat{k}_\mu = \frac{2}{a} \, \sin(\frac{k_\mu a}{2}) \qquad (7)$$

Since $\hat{k}_\mu = \frac{2}{a} \sin\left(\frac{k_\mu a}{2}\right)$ appears only to even powers and $\hat{k}_\mu^2 = \frac{2}{a^2}(1 - \cos(k_\mu a))$, we need only integrals of the type

$$\begin{aligned} F(q, n_1, n_2, n_3, n_4) &= \\ &= \lim_{\delta \to 0} \int dk \, \frac{\cos(k_1)^{n_1} \cos(k_2)^{n_2} \cos(k_3)^{n_3} \cos(k_4)^{n_4}}{\Delta_B^{(q+\delta)}} \end{aligned}$$

where

$$\Delta_B = 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4),$$

μ_B is the IR regularization mass, δ is an infinitesimal parameter for intermediate regularization.
$F(q; n_1, n_2, n_3, n_4)$ is symmetric in n_1, n_2, n_3, and n_4, $\Longrightarrow$
we consider only the case when $n_1 \geq n_2 \geq n_3 \geq n_4$.

if $n_4 \geq 2$ then $F(q,n_1,n_2,n_3,n_4) = F(q,n_1,n_2,n_3,n_4-2)\ -$

$$- \quad \frac{(n_4-1)F(q-1,n_1,n_2,n_3,n_4-1)-(n_4-2)F(q-1,n_1,n_2,n_3,n_4-3)}{q-1+\delta},$$

else if $n_3 \geq 2$ then $F(q,n_1,n_2,n_3,0) = F(q,n_1,n_2,n_3-2,0)\ -$

$$- \quad \frac{(n_3-1)F(q-1,n_1,n_2,n_3-1,0)-(n_3-2)F(q-1,n_1,n_2,n_3-3,0)}{q-1+\delta},$$

else if $n_2 \geq 2$ then $F(q,n_1,n_2,0,0) = F(q,n_1,n_2-2,0,0)\ -$

$$- \quad \frac{(n_2-1)F(q-1,n_1,n_2-1,0,0)-(n_2-2)F(q-1,n_1,n_2-3,0,0)}{q-1+\delta},$$

else if $n_1 \geq 2$ then $F(q,n_1,0,0,0) = F(q,n_1-2,0,0,0)\ -$

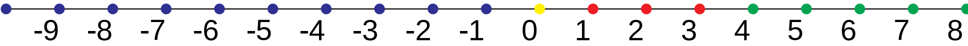
$$- \quad \frac{(n_1-1)F(q-1,n_1-1,0,0,0)-(n_1-2)F(q-1,n_1-3,0,0,0)}{q-1+\delta}.$$

$$
\begin{aligned}
F(q,n_1,n_2,n_3,1) &= (4+\mu_B^2)F(q,n_1,n_2,n_3,0) - F(q-1,n_1,n_2,n_3,0) \\
&- F(q,n_1+1,n_2,n_3,0) - F(q,n_1,n_2+1,n_3,0) \\
&- F(q,n_1,n_2,n_3+1,0), \\
F(q,n_1,n_2,1,0) &= \frac{1}{2}\Big((4+\mu_B^2)F(q,n_1,n_2,0,0) - F(q-1,n_1,n_2,0,0) \\
&- F(q,n_1+1,n_2,0,0) - F(q,n_1,n_2+1,0,0)\big), \\
F(q,n_1,1,0,0) &= \frac{1}{3}\Big((4+\mu_B^2)F(q,n_1,0,0,0) - F(q-1,n_1,0,0,0) \\
&- F(q,n_1+1,0,0,0)\big), \\
F(q,1,0,0,0) &= \frac{1}{4}\Big((4+\mu_B^2)F(q,0,0,0,0) - F(q-1,0,0,0,0)\Big).
\end{aligned}
$$

The above identities can be obtained using integration by parts.

Thus we obtain an expression for each integral $F(q, n_1, n_2, n_3, n_4)$ in terms of the functions

$$G_\delta(q, \mu_B^2) = \int \frac{dk}{(2\pi)^4} \frac{1}{(\Delta_B)^{q+\delta}},$$



- $q \geq 2$ – divergent parts:
  $G_\delta(q, \mu_B^2) = D(q, \mu_B^2) + J(q) + O(\mu_B^2) + O(\delta)$
- $q \leq 0$ – $O(\delta)$ terms:
  $G_\delta(q, \mu_B^2) = B(q) + \delta J(q) + O(\mu_B^2) + O(\delta^2)$

Computation of the Divergent Part (Fictitious Mass Regularization)

$$
\begin{aligned}
G_\delta(q,\mu_B^2) &= \frac{1}{\Gamma(q+\delta)}\int_0^\infty t^{q-1+\delta}\,dt\,\left[e^{-4t-\mu_B^2 t} I_0^4(t)\right]\\
&= \frac{1}{\Gamma(q+\delta)}\left\{\int_0^1 t^{q-1+\delta}\,dt\,\left[e^{-4t-\mu_B^2 t} I_0^4(t)\right]+\right.\\
&\quad +\int_1^\infty t^{q-1+\delta}\,dt\,e^{-\mu_B^2 t}\left[e^{-4t}I_0^4(t)-\frac{1}{(2\pi t)^2}\sum_{n=0}^{q-2}\frac{b_n}{t^n}\right]\\
&\quad \left.+\int_1^\infty t^{q-1+\delta}\,dt\,\frac{1}{(2\pi t)^2}\,e^{-\mu_B^2 t}\sum_{n=0}^{q-2}\frac{b_n}{t^n}\right\},
\end{aligned}
$$

$$
\begin{aligned}
J(q) = {} & \frac{1}{384(q-1)(q-2)^2(q-3)} \Big\{ \\
& 16(q-2)(q-3)\left[12 + 25(q-2)(q-3)\right] J(q-1) \\
& + 4(q-3)^2 \left[-17 - 35(q-3)^2\right] J(q-2) \\
& + 4 \left[1 + 5(q-3)^3(q-4) - 5(q-3)(q-4)^2\right] J(q-3) \\
& -(q-4)^4 J(q-4) \Big\} \\
& + \frac{1}{(q-2)} \left\{ D(q) - \frac{25}{24(q-1)} (2q-5) D(q-1) \right\} \\
& + \frac{1}{96(q-1)(q-2)^2} \Big\{ \left[17 + 105(q-3)^2\right] D(q-2) \\
& + \frac{5}{(q-3)} \left[-1 - 4(q-3)^2(q-4) + 2(q-4)^2\right] D(q-3) \\
& + \frac{5}{4(q-3)} (q-4)^3 D(q-4) \Big\};
\end{aligned}
$$

Thus integrals $G(q)$ can be expressed in terms of the quantities

$$\begin{aligned} Z_0 &= 0.154933390231060214084837208 \\ Z_1 &= 0.107781313539874001343391550 \\ F_0 &= 4.369225233874758 \end{aligned} \tag{8}$$

determined from the relations

$$\begin{aligned} F(1,0,0,0,0) &= 2Z_0 + O(\mu_B^2) \\ F(2,0,0,0,0) &= -\frac{l_C}{(2\pi)^2} + \frac{\bar{F}_0}{(2\pi)^2} + O(\mu_B^2) \\ F(3,0,0,0,0) &= \frac{1}{(2\pi)^2}\left(\frac{1}{2\,\mu_B^2} - \frac{l_C}{4} - \frac{13}{48} \right. \\ &\quad \left. + \frac{\bar{F}_0}{4}\right) - \frac{1}{128} + \frac{Z_1}{32} + O(\mu_B^2), \end{aligned} \tag{9}$$

where $\bar{F}_0 = F_0 - \ln 2$, $l_C = \ln \mu_B^2 + C$, $C = 0.577...$ is the Euler constant.

The operator of momentum in the one-dimensional quantum mechanics is well defined only on a continuous axis or segment, whereas to put it on a lattice presents a challenge. The problem is to construct a hermitian bilinear form of the type

$$(\varphi, \hat{p}_{latt}\psi)_{latt} \equiv \sum_{x,y} \bar{\varphi}_x P_{x,y} \psi_y \tag{10}$$

associated with

$$(\varphi, \hat{p}\psi)_{cont} = \int dx \; \bar{\varphi}(x) \frac{-\imath\, \partial}{\partial x} \psi(x), \tag{11}$$

- The form $(\varphi, \hat{p}_{latt}\psi)_{latt}$ must be hermitian
- The matrix $P_{x,y}$ must vanish at $x - y > na$, where n is some fixed number (locality).
- The form $(\varphi, \psi)_{latt}$ must go over to $(\varphi, \psi)_{cont}$ as the lattice spacing tends to zero, $a \to 0$.

The naive approximation of the derivative,

$$\partial\psi \to \frac{\psi(x+a) - \psi(x)}{a}$$

gives a nonhermitian operator of momentum with the eigenvalues

$$e^{ipa/2}\sin(pa/2)$$

To fulfil hermiticity, we can define $\hat{p}$ as the operator of multiplication:

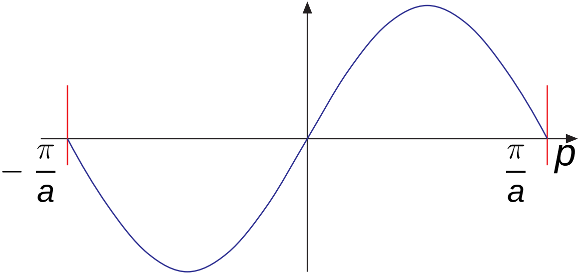
$$\hat{p}\tilde{\phi}(p) = \sin(pa/2)\tilde{\phi}(p); \tag{12}$$

thus we arrive at a NONLOCAL propagator
in the coordinate space.

The symmetric lattice approximation of the derivative

$$\partial\psi \to \frac{\psi(x+a) - \psi(x-a)}{2a}. \tag{13}$$

gives rise to the doubling problem: the form $(\psi, \hat{p}_{latt}\psi)_{latt}$ vaniches not only on the constant but also on **highly oscillating configurations**.



To obviate these difficulties, one can replace

$$\mathcal{L} = \bar{\psi}(i\hat{\partial} - m)\psi \qquad \longrightarrow$$

$$\mathcal{L}_W = \bar{\psi}(i\hat{\partial} - m)\psi - ra\bar{\psi}\partial_\mu\partial_\mu\psi$$

On a lattice, the Wilson action for free fermions has the form

$$S^F_{\text{FREE}} = \frac{a^3}{2} \sum_x \{\bar{\psi}_x(2ma + 8r)\psi_x \tag{14}$$

$$- \sum_{\mu=1}^{4} \left[\bar{\psi}_x(r - \gamma_\mu)\psi_{x+\hat{\mu}a} + \bar{\psi}_{x+\hat{\mu}a}(r + \gamma_\mu)\psi_x\right]\}$$

The respective fermion propagator is given by

$$\frac{m + \Delta + i \sum_{\mu=1}^{4} \gamma^{\mu} s_{\mu}(p)}{(m + \Delta)^2 + s^2(p)} \tag{15}$$

where

$$\begin{aligned} \Delta &= \frac{r}{a} \sum_{\mu=1}^{4} (1 - \cos(p_{\mu} a)), \\ s_{\mu}(p) &= \frac{1}{a} \sin(p_{\mu} a). \end{aligned} \tag{16}$$

r is the Wilson parameter.

Fermion Integrals

We consider integrals of the type

$$F(p, q; n_1, n_2, n_3, n_4) =$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{\cos^{n_1}(k_1)\cos^{n_2}(k_2)\cos^{n_3}(k_3)\cos^{n_4}(k_4)}{\Delta_B^q \Delta_F^{p+\delta}}$$

where

$$\begin{aligned} \Delta_F &= 10 - 4\sum_{\mu=1}^{4}\cos(k_\mu) \\ &+ \sum_{1\le\mu<\nu\le 4}\cos(k_\mu)\cos(k_\nu) + \mu_B^2 \end{aligned} \tag{17}$$

$$\Delta_B = 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4).$$

Making use of the recursion relations

$$
\begin{aligned}
&F(p,q,...,l,...) = F(p,q,...,l-2,...)\\
&+\mu_B^2\,(F(p,q,...,l-1,...) - F(p,q,...,l-3,...))\\
&-\ (F(p,q-1,...,l-1,...) - F(p,q-1,...,l-3,...))\\
&-q\ \frac{(F(p-1,q+1,...,l-1,...) - F(p-1,q+1,...,l-3,...))}{p-1+\delta}\\
&-\ \frac{((l-2)F(p-1,q,...,l-2,...) - (l-3)F(p-1,q,...,l-4,...))}{p-1+\delta}
\end{aligned}
$$

and

$$
\begin{aligned}
&F(p,q;n1,n2,n3,2) = \\
&F(p,q-2,n1,n2,n3,0) - 2\,\mu_B^2\,F(p,q-1,n1,n2,n3,0) \\
&-2\,F(p-1,q,n1,n2,n3,0) + (4+2\,\mu_B^2+\mu_B^4)\,F(p,q,n1,n2,n3,0) \\
&-F(p,q,n1+2,n2,n3,0) - F(p,q,n1,n2+2,n3,0) \\
&-F(p,q,n1,n2,n3+2,0), \\
&F(p,q,n1,n2,n3,1) = (\mu_B^2+4)\,F(p,q,n1,n2,n3,0) \\
&-F(p,q-1,n1,n2,n3,0) - F(p,q,n1+1,n2,n3,0) \\
&-F(p,q,n1,n2+1,n3,0) - F(p,q,n1,n2,n3+1,0)
\end{aligned}
$$

we can express $F(p,q;n_1,n_2,n_3,n_4)$ in terms of the integrals

$$
G(p,q) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{\Delta_B^q \Delta_F^{p+\delta}}
$$

$$F(p,q;1,1,1,1) =$$

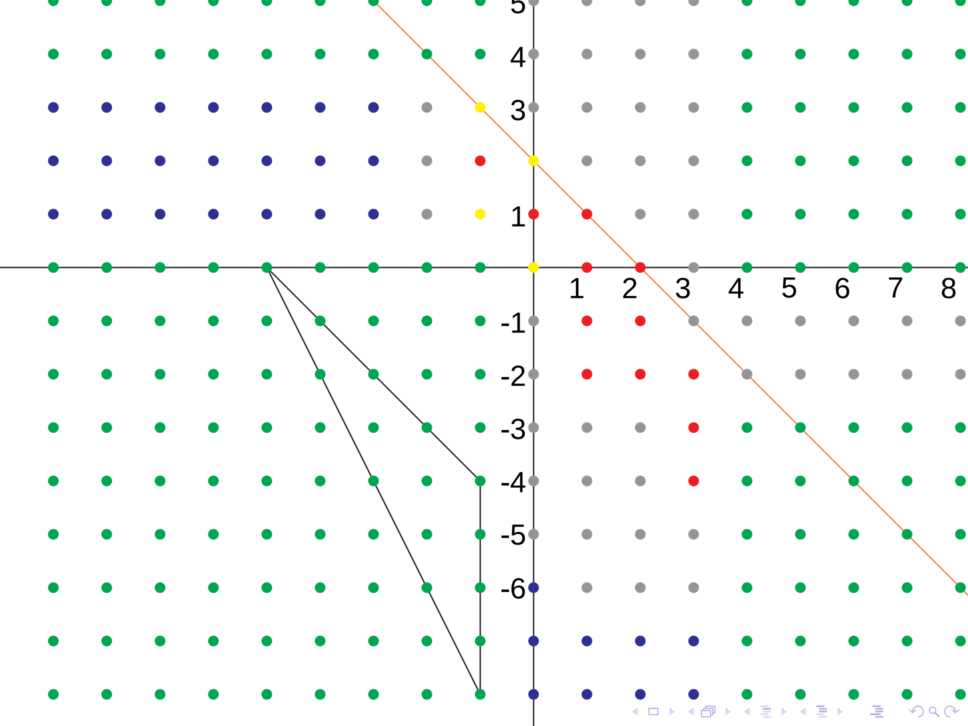
$$= \int \frac{d^4k}{(2\pi)^4} \frac{\cos(k_1)\cos(k_2)\cos(k_3)\cos(k_4)\{\,ZERO\}}{\Delta_B^q \Delta_F^{p+\delta}}$$

where

$$ZERO = -\Delta_F + 10 - 4\sum_{\mu=1}^{4}\cos(k_\mu) + \sum_{1\le\mu<\nu\le4}\cos(k_\mu)\cos(k_\nu) + \mu_B^2 \tag{18}$$

or

$$ZERO = -\Delta_B + 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4)$$



5
4
3
1
-1
-2
-3
-4
-5
-6
1
2
3
4
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6
7
8

It is convenient to represent $G_\delta(p,q)$ in the form

$$
\begin{aligned}
G_\delta(p,q) &= D(p,q;\mu_B^2) + B(p,q) + \delta\,(L(p,q;\mu_B^2) + J(p,q)) + O(\delta^2) \\
G_\delta(p,q) &= D(p,q;\mu_B^2) + J(p,q) + O(\delta), \qquad p > 0.
\end{aligned}
$$

$$
\begin{aligned}
J(-p,-q) = {} & (24J(-p+1,-q-1)(q+2)(q+3)(-5p+3) \\
& +2J(-p+1,-q-2)(6q^3-21(p-3)q^2 \\
& +q(-20p^2-91p+207)+12(-5p^2-7p+18)) \\
& +(p-1)(24J(-p+2,-q-2)(q+3)(-11p+14) \\
& +4J(-p+2,-q-3)(15q^2+q(-20p+121)+5(-6p^2+4p+35)) \\
& +J(-p+2,-q-4)(21q^2+8q(p+18)+2(-5p^2+26p+120))) \\
& +(p-1)(p-2)(24J(-p+3,-q-3)(-15p+31) \\
& +2J(-p+3,-q-4)(66q+73p+85) \\
& +4J(-p+3,-q-5)(10q+10p+23) \\
& +6J(-p+3,-q-6)q \\
& +6(p+3)J(-p+3,-q-6)) \\
& +(p-1)(p-2)(p-3)(192J(-p+4,-q-4) \\
& \qquad -20J(-p+4,-q-5)-3J(-p+4,-q-6)) \\
&)/24/(q+1)/(q+2)/(q+3)
\end{aligned}
$$

$$
\begin{aligned}
&+(24B(-p+1,-q-1)(q+2)(q+3)(-10p^3+39p^2-36p+3)\\
&+B(-p+1,-q-2)(q+3)((3(p-2)^2-1)(12q^2+6q-128)\\
&\qquad+(p-2)^2(-84q(p-2)-8(5(p-2)^2+54(p-2)+5)))\\
&+(p-1)^2(24B(-p+2,-q-2)(q+3)(-11p^2+28p-4)\\
&+B(-p+2,-q-3)(60q^2(2p-5)+4q(-20p^2+242p-485)\\
&\qquad+20(-26p^2+142p-199))\\
&+B(-p+2,-q-4)(21q^2(2p-5)+8q(p^2+36p-96)\\
&\qquad+2(p^2+300p-756)))\\
&+(p-1)^2(p-2)^2(-336B(-p+3,-q-3)\\
&\quad+4B(-p+3,-q-4)(33q+152)\\
&\quad+4B(-p+3,-q-5)(10q+53)+6B(-p+3,-q-6)(q+6))\\
&+24B(-p,-q)(q+1)(q+2)(q+3)(-3p^2+12p-11)\\
&)/24/(p-1)/(p-2)/(p-3)/(q+1)/(q+2)/(q+3);
\end{aligned}
$$

$G(p,q)$ can be expressed in therms of the quantities

$$Y_4 = \frac{J(1,0)}{2}, \qquad Y_5 = J(1,-1), \tag{20}$$

$$Y_6 = 2J(1,-2), \qquad Y_7 = \frac{J(2,-1)}{2},$$

$$Y_8 = J(2,-2), \qquad Y_9 = \frac{J(3,-2)}{2},$$

$$Y_{10} = J(3,-3), \qquad Y_{11} = 2J(3,-4),$$

$$Y_0 = \frac{J(2,0)}{4} - \frac{F_0}{16\pi^2}$$

and

$$
\begin{aligned}
Y_1 &= \frac{1}{48} - \frac{1}{4}\, Z_0 - \frac{1}{24}\, J(-1,2) + \frac{1}{12}\, J(0,1) + \frac{1}{12}\, J(1,0); \\
Y_2 &= \frac{1}{6} - \frac{1}{\pi^2} - Z_0 - \frac{1}{6}\, J(-1,2) + \\
&\quad \frac{1}{3}\, J(0,1) - \frac{1}{24}\, J(1,-2) - \frac{1}{12}\, J(1,-1) - \\
&\quad -\frac{17}{8}\, J(1,0) + 4\, J(1,1) - \frac{1}{48}\, J(2,-2) \\
&\quad +\frac{25}{6}\, J(2,-1) - 4\, J(2,0); \\
Y_3 &= -\frac{1}{384\pi^2} - F_0\, \frac{1}{128\pi^2} + \frac{1}{96}\, Z_0 - \\
&\quad \frac{1}{48}\, J(-1,3) + \frac{1}{192}\, J(0,1) + \frac{1}{48}\, J(0,2) + \frac{1}{48}\, J(1,1);
\end{aligned}
$$

Y_0	$-0.0184976584679165735 6$
Y_1	0.0037663633366186681 1
Y_2	0.0026539572948787935 4
Y_3	0.0002275154061514710 7
$Y_4 = \mathcal{F}(1,0)$	0.0853903635953206791 4
$Y_5 = \mathcal{F}(1,-1)$	0.4693633100269961447 5
$Y_6 = \mathcal{F}(1,-2)$	3.3945690736771300058 6
$Y_7 = \mathcal{F}(2,-1)$	0.0518801950390113663 6
$Y_8 = \mathcal{F}(2,-2)$	0.2387477375634147852 0
$Y_9 = \mathcal{F}(3,-2)$	0.0344764414380322314 5
$Y_{10} = \mathcal{F}(3,-3)$	0.1320272712278129308 5
$Y_{11} = \mathcal{F}(3,-4)$	0.7516719903029568225 4

Table: New constants appearing in the general fermionic case.

The boson lattice propagator in the coordinate space has the form

$$
\begin{aligned}
G_B(x_1,...,x_4) &= \int \frac{dk}{(2\pi)^4}\frac{e^{ik_1x_1}\ e^{ik_2x_2}\ e^{ik_3x_3}\ e^{ik_4x_4}}{\Delta_B} \\
&= \int \frac{dk}{(2\pi)^4}\frac{\cos(k_1x_1)\ ...\ \cos(k_4x_4)}{\Delta_B}
\end{aligned}
$$

$$
\Delta_B = 4 + \mu_B^2 - \cos(k_1) - \cos(k_2) - \cos(k_3) - \cos(k_4).
$$

$$
G_B(x_1,...,x_4) \simeq \sum F(1;n_1,n_2,n_3,n_4) \qquad (Z_0,Z_1)
$$